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Zig zag symmetry in AdS/CFT duality

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Abstract. The validity of the Bianchi identity, which is intimately connected with the zig zag symmetry, is established, for piecewise continuous contours, in the context of Polakov's gauge field–string connection in the large 't Hooft coupling limit, according to which the chromoelectric 'string' propagates in five dimensions with its ends attached on a Wilson loop in four dimensions. An explicit check in the wavy line approximation is presented.

1 Introduction

The employment of string theoretical methods to build inroads to QCD, especially at the non-perturbative level, is a problem that has been posed by Polyakov [1] over two and a half decades ago. Since then, string theory has made notable advancements in this regard, both as regards applications to high energy processes [2, 3] and in the direction of expediting high order, perturbative computations; see, e.g. [4] for a review presentation, wherein relevant aspects to collider physics applications are also discussed; for recent advances on this subject, see [5].

In an independent development and in the context of 't Hooft's [6] large N, $\lambda \equiv g^2_{\mathrm{YM}} N \gg 1$ limit, Polyakov [7] proposed, in an attempt to capture the essential characteristics of a string relevant to QCD and one which accommodates the Liouville mode, a setting according to which the string appropriate for representing the chromoelectric flux lines of a pure Yang–Mills theory must propagate in a 5-dimensionalal environment the metric of which reads

$$\mathrm{d}s^2 = a(y)\big(\mathrm{d}y^2 + \mathrm{d}x^2_\mu\big)\,, \quad a(y) \sim y^{-2}(y \to 0)\,, \tag{1}$$

with the gauge theory 'living' at the boundary, $y = 0$, of this space. The above description will contain additional dimensions, if the 4D theory has extra matter fields, as happens in the AdS/CFT case [8]. The requirement of conformal symmetry fixes

$$\mathrm{d}s^2 = a(y) = \frac{R^2}{y^2}\,, \quad R^2 = \alpha'\sqrt{\lambda}. \tag{2}$$

The Wilson loop functional [9]

$$W[C] = \frac{1}{N}\Big\langle \mathrm{Tr}\, P \exp \mathrm{i} \oint_C A_\mu \,\mathrm{d}x_\mu \Big\rangle_A \tag{3}$$

plays a basic role in the gauge–string correspondence in Polyakov's scheme, wherein the open string propagating in a 5-dimensional background (2) has its two ends attached onto a loop contour. The latter, as already mentioned, lives in four dimensions.

The working assumption for quantifying such a proposal is that, in the large λ limit, the Wilson loop functional is expected to behave as

$$W[C] \propto \mathrm{e}^{-\sqrt{\lambda}\mathcal{A}_{\min}(C)}\,, \tag{4}$$

where $\mathcal{A}_{\min}$ is the minimal area swept by the string and bounded by the contour C. This statement constitutes a zeroth, WKB-type, approximation to the problem.

Now, the loop casting of QCD has a long history, which is intimately associated with theoretical efforts to probe its non-perturbative content. It constitutes a well defined strategy of formulating QCD and enjoys, in its discrete version, universal acceptance as *the* methodology for investigating non-perturbative issues surrounding strong force dynamics.

A corresponding, direct continuum casting of QCD, based on the Wilson functional, gives rise to the loop equation formalism that has been extensively pursued by Makeenko and Migdal [10–12], as well as by Polyakov in [1], and that has provided a multitude of powerful insights to the theory. Within the framework of this scheme, a property of vital importance Wilson functionals must possess is that of zig zag, or equivalently backtracking, invariance. The same symmetry plays a fundamental role in Polyakov's choice of the background (2) that accommodates the fluctuations of the random surfaces bounded by the contour C. Such a requirement characterizes, in general, the so-called Stokes-type functionals whose basic property is, precisely, that they do not change when a small path passing back and forth is added to any smooth section of the loop at any given point. In mathematics, this prop-

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erty is associated with what are known as Chen integrals. Quantitatively speaking, the backtracking invariance in the loop formalism assumes the form (see, e.g., [11, 12])

$$\epsilon^{\kappa\lambda\mu\nu}\partial_\lambda^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W[C] = 0\,, \tag{5}$$

with $\delta\sigma_{\mu\nu}$ and ∂_λ^x the surface and path derivatives whose action will be specified later. From the point of view of QCD the relevance of Stokes-type functionals is traced to the fact that they facilitate the proof of the non-abelian Stokes theorem; hence their name.

In order to establish the validity of the non-abelian Stokes theorem in the loop formalism of QCD the key role is played by the Bianchi identity, which assures the commutativity of differentiations performed on a Wilson loop, in a surface independent manner [11–14]. In fact, one easily verifies that

$$\begin{aligned}\epsilon^{\kappa\lambda\mu\nu}\partial_\lambda^x \frac{\delta}{\delta\sigma_{\mu\nu}} W &= \frac{1}{N}\epsilon^{\kappa\lambda\mu\nu}\,\mathrm{Tr}\,P\left\langle \nabla_\lambda F_{\mu\nu} \exp \mathrm{i}\oint_C A_\mu\,\mathrm{d}x_\mu \right\rangle_A \\ &= 0\,. \end{aligned} \tag{6}$$

Demonstrating the validity of the Bianchi identity, equivalently zig zag invariance, within the framework of the field–string connection according to the proposal in [7], is the central objective of this work. More specifically, the stated objective of this paper is to establish that

$$\epsilon^{\kappa\lambda\mu\nu}\partial_\lambda^x \frac{\delta}{\delta\sigma_{\mu\nu}} \exp(-\sqrt{\lambda}A_{\min}) \approx 0\,,$$

in the limit $\lambda \to \infty$.

Our exposition is organized as follows. In the next section we introduce the area derivative operator appropriate for acting on the Wilson loop functional. To begin with, on the field theoretical side it is through this action that one establishes the loop equations. On the string side, it will turn out that it plays a key role in establishing the Bianchi identity. The variational analysis for the verification of both the loop equations and the Bianchi identity will be greatly facilitated by employing a methodology, developed in [15, 16], which directly addresses a situation involving a surface bounded by a closed contour in four (D) dimensions that variationally protrudes into five ($D+1$) dimensions. This approach will be reviewed in Sect. 2, where the all important quantity, to be designated as the **g**-function, will emerge. This quantity, as it turns out, contains all the dynamics in the advocated approach. The area derivative operator will also be introduced in this section and some realizations of a general nature will be made regarding its action on the Wilson loop functional. The next section (Sect. 3) is devoted to the study of the normal variations, with respect to the boundary of the Wilson loop, of the **g**-function. These variations will play a pivotal role in our subsequent quantitative considerations. In Sect. 4 we apply the mathematical formalism developed to this point to verify, on the string side, the loop equation of Makeenko and Migdal [10]. At the same time we shall derive a result, conditional, at this stage, concerning the Bianchi identity. The conditionality of the result will be attributed to the fact that the vector basis adopted to describe the 5-dimensional surface spanned by the string is too general to control the precise manner by which it "collapses" onto the corresponding 4-dimensional Wilson loop configuration. Accordingly, only a *condition* for the validity of the Bianchi identity can be obtained. Full confirmation becomes precise in Sect. 5, where a certain Wilson contour of sufficient generality introduced in [15] and characterized as 'wavy line' configuration, is employed to rigorously demonstrate the validity of the Bianchi identity. Some general, concluding comments are presented in the final section.

2 String action functional and the area derivative operator

In this section we present the general form of the area derivative operator, which is to act on a Wilson loop configuration. We begin our discussion by presenting a condensed summary of the setting promoted in [15, 16], which is nicely suited for conducting analytical considerations pertaining to the proposal of [7]. The relevant string action functional according to this reference is (Euclidean formalism employed throughout)

$$\begin{aligned} S[\mathbf{x}(\xi), y(\xi)] &= \frac{1}{2}\sqrt{\lambda}\int_D \mathrm{d}^2\xi G_{MN}(x(\xi))\partial_a x^M(\xi)\partial_a x^N(\xi) \\ &= \frac{1}{2}\sqrt{\lambda}\int_D \frac{\mathrm{d}^2\xi}{y^2(\xi)}\left[(\partial_a \mathbf{x}(\xi))^2 + (\partial_a y(\xi))^2\right]\,, \end{aligned} \tag{7}$$

where $x^M = (y, \mathbf{x}) = (y, x^\mu)$, $M, N = 0, 1, \ldots, D$; $\mu = 1, \ldots, D$, with the y-coordinate taking a zero value at the boundary and growing toward infinity as one moves deeper into the interior of the AdS_5 space.[1]

In [15, 16] a mathematical machinery was developed for the purpose of studying loop dynamics in reference to the above action functional. We shall adopt the strategy introduced in these works, the immediate aim being to determine the action of the area derivative operator [17]

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x(\sigma))} = \lim_{\eta\to 0}\int_{-\eta}^{\eta}\mathrm{d}h\, h \frac{\delta^2}{\delta x_\mu\left(\sigma+\frac{h}{2}\right)\delta x_\nu\left(\sigma-\frac{h}{2}\right)} \tag{8}$$

on a piecewise regular Wilson loop contour.

The loop functional is to be minimized under the boundary conditions $\mathbf{x}|_{\partial D} = \mathbf{c}(\alpha(\sigma))$ and $y|_{\partial D} = 0$, with the parametrization chosen so that

$$A_{\min}[\mathbf{c}(\sigma)] = \min_{\{\alpha(\sigma)\}}\min_{\{\mathbf{x},y\}} S[\mathbf{x}(\xi), y(\xi)]\,. \tag{9}$$

The functional $A_{\min}$ is invariant under reparametrizations of the boundary, a property that can be easily deduced

[1] To connect, in a general sense, our present work with the AdS/CFT conjecture [8], we shall, in a loose sense, refer to the 5-dimensional space-time background of Polyakov's scheme, wherein conformal invariance is implicitly assumed, as AdS_5.

from the above minimization condition ($c'_\mu(s) = \frac{\mathrm{d}}{\mathrm{d}s}c_\mu(s)$):

$$c'_\mu(\sigma)\frac{\delta A_{\min}}{\delta c_\mu(\sigma)} = 0\,. \tag{10}$$

Following [15, 16], we adopt the static gauge $y(t,\sigma) = t$ and place the loop on the boundary of the AdS_5 space, i.e. we set $t = 0$. One accordingly writes

$$\mathbf{x}(t,\sigma) = \mathbf{c}(\sigma) + \frac{1}{2}\mathbf{f}(\sigma)t^2 + \frac{1}{3}\mathbf{g}(\sigma)t^3 + \frac{1}{4}\mathbf{h}(\sigma)t^4 + \dots\,, \tag{11}$$

where, for now, the curve $\mathbf{c}(\sigma)$ is assumed to be differentiable everywhere. If there are cusps on the loop contour (i.e., discontinuities in the first derivative) the above expansion must be understood piecewise. Surface minimization leads to the elimination of the linear term in the expansion and determines its next coefficient:

$$\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\mathbf{c}'}{\mathbf{c}'^2}\,. \tag{12}$$

The coefficient $\mathbf{g}(\sigma)$ is, at this point, unspecified. Imposition of the Virasoro constraints leads to

$$\mathbf{c}'\cdot\mathbf{g} = 0\,. \tag{13}$$

It turns out that the latter relation simply expresses the reparametrization invariance of the minimal area (9); and, hence, the quantity $\mathbf{g}(\sigma)$, to be referred to as the $\mathbf{g}$-function from now on, remains undetermined. More illuminating, for our purposes, is an interim result through which (13) is derived and which reads as follows:

$$\frac{\delta A_{\min}}{\delta\mathbf{c}(\sigma)} = -\sqrt{\mathbf{c}'^2}\mathbf{g}(\sigma)\,. \tag{14}$$

The above relation underlines the dynamical significance of the $\mathbf{g}$-function: it provides a measure of the change of $A_{\min}$ when the Wilson loop contour is altered as a result of some interaction that reshapes its geometrical profile.

Consider, now, the action of the area derivative on the Wilson loop functional:

$$\begin{aligned}&\frac{\delta}{\delta\sigma_{\mu\nu}(\sigma)}W[C]\\ &= \lim_{\eta\to 0}\int_{-\eta}^{\eta}\mathrm{d}h\,h\left[-\sqrt{\lambda}\frac{\delta^2 A_{\min}}{\delta c_\mu\left(\sigma+\frac{h}{2}\right)\delta c_\nu\left(\sigma-\frac{h}{2}\right)}\right.\\ &\qquad\left.+\lambda\frac{\delta A_{\min}}{\delta c_\mu\left(\sigma+\frac{h}{2}\right)}\frac{\delta A_{\min}}{\delta c_\nu\left(\sigma-\frac{h}{2}\right)}\right]W[C]\,.\end{aligned} \tag{15}$$

As is known [18], the area derivative is a well defined operation only for smooth contours, i.e. ones that are differentiable everywhere. In such a case the last term in the above equation gives a zero contribution. If the loop under consideration has cusps, as happens in the framework of non-trivial situations, the operation must be understood piecewise; see [19] for such a realization.

To further facilitate our considerations we follow [15, 16] by choosing the coordinate σ on the minimal surface such that

$$\mathbf{c}'^2(\sigma) = 1\,,\quad \dot{\mathbf{x}}(t,\sigma)\cdot\mathbf{c}'(\sigma) = 0\,.$$

We also introduce an orthonormal basis, which adjusts itself along the tangential ($\mathbf{t}$) and normal ($\mathbf{n}^a$, $a = 1,\dots,D-1$) directions defined by the contour, as follows:

$$\begin{gathered}\{\mathbf{t},\mathbf{n}^a\}\,,\quad a = 1,\dots,D-1\,,\\ \mathbf{t} = \frac{\mathbf{c}'}{\sqrt{\mathbf{c}^2}}\,,\quad \mathbf{n}^a\cdot\mathbf{t} = 0\,,\quad \mathbf{n}^a\cdot\mathbf{n}^b = \delta^{ab}\,.\end{gathered} \tag{16}$$

We now write

$$\frac{\delta}{\delta c_\mu} = n^a_\mu\left(\mathbf{n}^a\cdot\frac{\delta}{\delta\mathbf{c}}\right) + t_\mu\left(\mathbf{t}\cdot\frac{\delta}{\delta\mathbf{c}}\right) \equiv n^a_\mu\frac{\delta}{\delta\mathbf{n}^a} + t_\mu\frac{\delta}{\delta\mathbf{t}}\,, \tag{17}$$

and upon using (12) and (13), as well as setting $s = \sigma + h/2$ and $s' = \sigma - h/2$, we determine

$$\begin{aligned}&\frac{\delta^2 A_{\min}}{\delta c_\mu(s)\delta c_\nu(s')}\\ &= -\frac{\delta g^a(s)}{\delta\mathbf{n}^b(s')}n^a_\mu(s)n^b_\nu(s') + R_{\mu\nu}(s,s')\delta'(s-s')\,,\end{aligned} \tag{18}$$

where

$$\begin{aligned}R_{\mu\nu}(s,s') = {}& 2\mathbf{g}(s)\cdot\mathbf{n}^a(s')t_\mu(s)n^a_\nu(s')\\ &+\mathbf{g}(s)\cdot\mathbf{t}(s')t_\mu(s)t_\nu(s')\\ &-\mathbf{t}(s)\cdot\mathbf{n}^a(s')g_\mu(s)n^a_\nu(s')\,.\end{aligned} \tag{19}$$

From the defining expression, see (8), one immediately realizes that only terms $\sim\delta'(s-s')$ in an antisymmetric combination $R_{[\mu\nu]}$ will give non-zero contributions to the area derivative. It, thus, becomes obvious that the last term in (18) produces the result

$$R_{[\mu\nu]}(\sigma,\sigma) = t_{[\mu}(\sigma)g_{\nu]}(\sigma)\,. \tag{20}$$

Turning our attention to the first term on the r.h.s. of (18) we note that non-vanishing contributions should have the form

$$(r^a q^b - r^b q^a)n^a_\mu n^b_\nu\delta'(s-s')\,, \tag{21}$$

where $r^a = \mathbf{n}^a\cdot\mathbf{r}$ and $q^a = \mathbf{n}^a\cdot\mathbf{q}$. These functions must be determined from the coefficients of the expansion (11); otherwise the above contribution would be contour independent, having no impact on a calculation associated with non-trivial dynamics. In conclusion, a simple qualitative analysis, based on the scale invariance of $A_{\min}$, indicates that a contribution of the type (21) does not exist. This qualitative observation can be further substantiated through a straightforward argument based on dimensional grounds. Indeed, from (11) it can be observed that under a change of scale of the form $\mathbf{c}\to\lambda\mathbf{c}$, $(t,\sigma)\to(\lambda t,\lambda\sigma)$ one has

$$\mathbf{c}'\to\mathbf{c}'\,,\quad \mathbf{f}\to\frac{1}{\lambda}\mathbf{f}\,,\quad \mathbf{g}\to\frac{1}{\lambda^2}\mathbf{g}\,,\quad \dots$$

On the other hand, now, the area derivative, being of second order, should scale $\sim \frac{1}{\lambda^2}$. In turn, this means that one of the quantities $\mathbf{r}$ or $\mathbf{q}$, which must arise through transverse variations of $\mathbf{g}$, should be aligned with the tangential vector $\mathbf{t}$, which, by definition, has zero transverse components. Thus, the only antisymmetric combination with the right scaling behavior must be either of the form $r^a f'^b - r^b f'^a$, or $r^a g^b - r^b g^a$, where $r^a \sim n_i^a c_i'$, with $i = 2, \dots$ But such expressions must be excluded, since they pick out a certain direction in the 4-dimensional space, whereas the area derivative must be a second rank tensor.

Referring to the formula for the area derivative, one immediately surmises that the first term on the r.h.s. of (18) gives a null contribution, since the antisymmetric term is proportional to $\delta(s-s')$, and *not* $\delta'(s-s')$. We have, therefore, determined that

$$\lim_{\eta\to 0}\int_{-\eta}^{\eta} \mathrm{d}h\, h \frac{\delta^2 A_{\min}}{\delta c_\mu(\sigma+\frac{h}{2})\delta c_\nu(\sigma-\frac{h}{2})} = t_{[\mu}(\sigma) g_{\nu]}(\sigma)\,. \quad (22)$$

In order to check the validity of the Bianchi identity we need a quantitative expression of the, with respect to the 4- (D-) dimensional surface of the Wilson loop, normal variations of the $\mathbf{g}$-function. It will turn out that the antisymmetric part of the variations will play a determining role in the derivation of the Bianchi identity. A quantitative study of these normal deviations will be conducted in the next section and the relevant results will further justify the line of arguments promoted in this section.

3 The normal variation of the g-function

We start the considerations in this section by remarking that the path derivative entering the Bianchi identity can be defined by [11, 12]

$$\partial_\mu^{c(s)} = \lim_{\epsilon\to 0}\int_{s-\epsilon}^{s+\epsilon} \mathrm{d}s' \frac{\delta}{\delta c_\mu(s')}\,. \quad (23)$$

Accordingly, as becomes obvious from (22) in the previous section, one needs an explicit expression for the normal variations of the $\mathbf{g}$-function. In fact, their antisymmetric part, it will turn out, will play a decisive role concerning the eventual derivation of the Bianchi identity concerned, as will be explicitly established in the sections to follow.

Let us introduce at every point of the surface bounded by the loop a basis $\{n_M^a(t,s)\}$ of $D-1$ orthonormal vectors that satisfy the conditions

$$n_M^a(t,s)\dot{x}_M(t,s) = n_M^a(t,s)x'_M(t,s) = 0\,, \quad (24)$$

where $G_{MN} n_M^a n_N^b = \delta^{ab}$ and $n_\mu^a(0,s) = n_\mu^a(s)$.

Under the normal variation

$$x_M(t,s) \to x_M(t,s) + \psi_M(t,s)\,,$$
$$\psi_M(t,s) = \phi^a(t,s) n_M^a(t,s)\,, \quad (25)$$

the change of the minimal surface to second order in ϕ^a reads

$$S^{(2)} = \int \mathrm{d}^2\xi \big[\sqrt{g}(g^{\alpha\beta}\partial_\alpha\psi^a\partial_\beta\psi^a + 2g^{\alpha\beta}\omega_\alpha^{[ab]}\partial_\beta\psi^a\psi^b$$
$$+ 2\psi^a\psi^a) + O(t^2\psi^2)\big]\,, \quad (26)$$

where we have written $\psi^a \equiv t\phi^a$ and we have introduced $g_{\alpha\beta} = G_{MN}\partial_\alpha x_M \partial_\beta x_N$, while the antisymmetric quantities $\omega_\alpha^{[ab]}$ are the spin connection coefficients, given by

$$\omega_\alpha^{[ab]} = \partial_\alpha n_M^a \cdot n_M^a\,. \quad (27)$$

Details of the analysis can be found in [16]. Here, all we need is the third order term in an expansion of ψ_M in powers of t. Notice that by taking into account that ϕ is regular as $t \to 0$, we have omitted terms $\sim t^4$ in (26) that do not contribute to the normal variation of the $\mathbf{g}$-function.

Using the expansion (11) one easily determines

$$g_{\alpha\beta} = \frac{1}{t^2}\begin{pmatrix} 1 + \mathbf{f}^2 t^2 + 2\mathbf{f}\cdot\mathbf{g}t^3 & \frac{1}{2}\mathbf{f}\cdot\mathbf{f}' t^3 \\ \frac{1}{2}\mathbf{f}\cdot\mathbf{f}' t^3 & 1 - \frac{1}{2}\mathbf{f}^2 t^2 - \frac{2}{3}\mathbf{f}\cdot\mathbf{g}t^3 + O(t^2) \end{pmatrix} \quad (28)$$

and

$$\sqrt{g} = \frac{1}{t^2}\left(1 + \frac{2}{3}\mathbf{f}\cdot\mathbf{g}t^3\right) + O(t^2)\,. \quad (29)$$

Now, the area derivative receives contributions from antisymmetric terms. We, therefore, have to find the behavior of the spin connection as $t \to 0$. This cannot be done in a unique way if $D > 2$. What one can do is to expand the basis vectors $n_M^a(t,s)$ as a power series in t:

$$n_0^a(t,s) = tk_0^a(s) + \frac{1}{2}t^2 l_0^a(s) + \frac{1}{3}t^3 m_0^a(s) + \dots\,,$$
$$\mathbf{n}^a(t,s) = t\mathbf{k}^a(s) + \frac{1}{2}t^2\mathbf{l}^a(s) + \frac{1}{3}t^3\mathbf{m}^a(s) + \dots \quad (30)$$

Combining these relations with (24) and using the expansion (11) we can determine

$$k_0^a = f^a\,, \quad l_0^a = -2(\mathbf{k}^a\cdot\mathbf{f} + g^a)\,,$$
$$m_0^a = -3\left(\frac{1}{2}\mathbf{l}^a\cdot\mathbf{f} + \mathbf{k}^a\cdot\mathbf{g} + h^a\right) \quad (31)$$

and

$$\mathbf{k}^a\cdot\mathbf{c}' = 0\,, \quad \mathbf{l}^a\cdot\mathbf{c}' + f'^a = 0\,, \quad \mathbf{m}^a\cdot\mathbf{c}' + g'^a + \frac{3}{2}\mathbf{k}^a\cdot\mathbf{f} = 0\,. \quad (32)$$

From the orthonormality condition we find that

$$\mathbf{k}^a\cdot\mathbf{n}^b(s) + \mathbf{k}^b\cdot\mathbf{n}^a(s) = 0\,,$$
$$2k_M^a\cdot k_M^b + \mathbf{l}^a\cdot\mathbf{n}^b(s) + \mathbf{l}^b\cdot\mathbf{n}^a(s) = 0\,,$$
$$\frac{3}{2}l_M^a\cdot l_M^b + \mathbf{m}^a\cdot\mathbf{n}^b(s) + \mathbf{m}^b\cdot\mathbf{n}^a(s) = 0\,. \quad (33)$$

With the above in place we return to our central objective and, to start with, assume that

$$\mathbf{k}^a\cdot\mathbf{c}' = 0 \to \mathbf{k}^a = \mathbf{0}\,, \quad (34)$$

which means that

$$\mathbf{l}^a \cdot \mathbf{c}' = -f'^a ,$$
$$\mathbf{l}^a \cdot \mathbf{n}^b(s) + \mathbf{l}^b \cdot \mathbf{n}^a(s) = -2k_0^a k_0^b = -2f^a f^b . \tag{35}$$

From these relations we conclude that

$$\mathbf{l}^a = -f'^a \mathbf{c}' - f^a \mathbf{f} + \Lambda^{ab}\mathbf{n}^b(s) ,$$
$$\mathbf{m}^a = -g'^a \mathbf{c}' - \frac{3}{2}(g^a \mathbf{f} + f^a \mathbf{g}) + M^{ab}\mathbf{n}^b(s) , \tag{36}$$

with Λ^{ab} and M^{ab} antisymmetric, but otherwise arbitrary.

The first one, Λ^{ab}, enters the second order term in the expansion (30) and consequently contributes to the normal variation of the $\mathbf{g}$-function and through it to the area derivative. The observation here is that this function cannot be exclusively determined from the functions $\mathbf{c}'$, $\mathbf{f}$, $\mathbf{g}, \dots$ which, in turn, determine $A_{\min}$. This can be deduced, through scaling properties as follows: under a change of scale $\mathbf{c} \to \lambda \mathbf{c}$, $(t,s) \to \lambda(t,s)$, it must behave as $\Lambda \to \frac{1}{\lambda^2}\Lambda$, as can be seen from (30). Taking, now, into account that $\mathbf{c}' \to \mathbf{c}'$, $\mathbf{f} \to \frac{1}{\lambda}\mathbf{f}$, $\mathbf{g} \to \frac{1}{\lambda^2}\mathbf{g}, \dots$ and that $\mathbf{n}^a(s)\cdot\mathbf{c}' = 0 \to c'^a = 0$, it becomes obvious that it is impossible to find an antisymmetric combination of the coefficient functions with the correct scaling behavior. The same reasoning, in fact, justifies, a posteriori, (34). The remaining possibilities are $\Lambda^{ab} = r^a g^b - r^b g^a$ or $\Lambda^{ab} = r^a f'^b - r^b f'^a$, with $r^a = n_i^a c_i'$, $i = 2, \dots, D$. But these are excluded because the produced l_μ^a are not 4-dimensional vectors. The second quantity, M^{ab}, must scale as $M^{ab} \to \frac{1}{\lambda^3}M^{ab}$ and consequently $M^{ab} \sim g^a f^b - g^b f^a$. Through this analysis the basis vectors are determined as follows:

$$n_0^a(t,s) = -tf^a - t^2 g^a - t^3(h^a - f^a\mathbf{f}') + \mathcal{O}(t^4) ,$$
$$\begin{aligned}\mathbf{n}^a(t,s) &= \mathbf{n}^a(s) - \frac{1}{2}t^3(g^a\mathbf{f} + f^a\mathbf{g}) \\ &\quad + \frac{2}{3}t^3\left(g'^a\mathbf{f} + f^a\mathbf{g} + \frac{2}{3}g'^a\mathbf{c}'\right) \\ &= \frac{1}{3}t^3\mathbf{n}^a M^{ab} + \mathcal{O}(t^4) .\end{aligned} \tag{37}$$

For the behavior of the spin connection we also need the derivative $\mathbf{n}'^a(s)$. What we know about it comes from the orthonormality condition

$$\mathbf{n}^a(s)\cdot\mathbf{c}' = 0 \to -\mathbf{n}'^a(s)\cdot\mathbf{c} = -\mathbf{n}^a(s)\cdot\mathbf{c}''(s) = -\mathbf{c}''^a(s) . \tag{38}$$

Adopting the same arguments as before we conclude from the preceding relation that

$$\mathbf{n}'^a(s) = -(\mathbf{n}^a(s)\cdot\mathbf{c}'')\mathbf{c}' = -\mathbf{c}''^a\mathbf{c}' . \tag{39}$$

In conclusion, through the above analysis we have determined that

$$\omega_t^{[ab]} = \frac{1}{2}t^2\kappa_0(g^a f^b - g^b f^a) \equiv \frac{1}{2}t^2 r^{ab} , \quad \omega_s^{[ab]} = \mathcal{O}(t^3) , \tag{40}$$

with the constant κ_0 remaining undetermined at the present level of the calculation.

Knowing the behavior of all the terms we now return to (26) and demand the perturbed surface also to be minimal. This leads to the equation

$$\partial_\beta\left(\sqrt{g}g^{\alpha\beta}\partial_\alpha\psi^a\right) - 2\sqrt{g}\psi^a + 2\sqrt{g}g^{\alpha\beta}\omega_\alpha^{[ab]}\partial_\beta\psi^b = \mathcal{O}(t^2\psi) . \tag{41}$$

To solve this equation we start from its asymptotic form as $t \to 0$, treating the other terms as small perturbations. At this point it becomes very convenient to introduce [15, 16] the Fourier transform

$$\phi^a(t,s) = \phi^a(t,s'+h) = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}ph}\tilde{\phi}^a(t,p) , \tag{42}$$

with $s = \sigma + \frac{h}{2}$ and $s' = \sigma' - \frac{h}{2}$, the point at which the area derivative is applied. The relevant observation here is that one is interested in large values for the variable $p \sim \frac{1}{h}$, since the variable h is integrated in the vicinity of zero; cf. (8).

On the other hand, one can be convinced, by appealing to (41), that the values of t that are involved in our analysis are $t \sim \frac{1}{|p|} \sim h$. With these estimations (40) can be rewritten by retaining only those terms that are relevant to the normal variation of the $\mathbf{g}$-function. To accomplish this task the coefficient functions must be expanded around the point s'. The general form of such an expansion can be read from

$$\begin{aligned}F(s) &= F(s') + (s-s')F'(s') + \dots \\ &= F(s') + hF'(s') + \dots ,\\ h\phi^a(t,s) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}ph}h\tilde{\phi}^a(t,p) \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}ph}\mathrm{i}\partial_p\tilde{\phi}^a(t,p) .\end{aligned} \tag{43}$$

Given the above, (40) reads, in Fourier space,

$$\hat{L}_4^{ab}(t,p)\tilde{\phi}^b(t,p) = \hat{L}_2^{ab}(t,p)\tilde{\phi}^b(t,p) + \hat{L}_1^{ab}(t,p)\tilde{\phi}^b(t,p) + \dots , \tag{44}$$

where we have written

$$\hat{L}_4^{ab} \equiv \left(\frac{1}{t^2}\partial_t^2 - \frac{2}{t}\partial_t - \frac{p^2}{t^2}\right)\delta^{ab} , \quad \hat{L}_2^{ab} \equiv \mathbf{f}^2\left(\partial_t^2 + p^2\right)\delta^{ab} .$$
$$\begin{aligned}\hat{L}_1^{ab} \equiv \Bigg\{&\left[2\mathbf{f}\cdot\mathbf{f}'\mathrm{i}\partial_p + \frac{4}{3}t(\mathbf{f}\cdot\mathbf{g})\right]\left(\partial_t^2 + p^2\right) + \frac{4}{3}\mathbf{f}\cdot\mathbf{g}\partial_t \\ &- \frac{3}{2}\mathbf{f}\cdot\mathbf{f}'\mathrm{i}p + t\mathbf{f}\cdot\mathbf{f}'\mathrm{i}p\partial_t\Bigg\}\delta^{ab} + r^{ab}\left(\frac{1}{t} - \partial_t\right) .\end{aligned} \tag{45}$$

The subscripts labeling the operators in the above relation serve to signify their asymptotic behavior as $|p| \to \infty$:

$$\hat{L}_4^{ab}\tilde{\phi}^b \sim O(p^4) , \quad \hat{L}_2^{ab}\tilde{\phi}^b \sim O(p^2) , \quad \hat{L}_1^{ab}\tilde{\phi}^b \sim O(p) . \tag{46}$$

The neglected terms in (44) are of order $\mathcal{O}(p^0)$, so their contribution will be four times weaker than the strongest one and thus will be irrelevant as far as we are interested in the normal variation of the $\mathbf{g}$-function.

The solution of (44) can be written as

$$\tilde\phi^a(t,p)=\tilde\phi^a_{(0)}(t,p) + \int_0^\infty \mathrm{d}t' G_p(t,t')\big[\hat L_2^{ab}(t',p)+\hat L_1^{ab}(t',p)\big]\tilde\phi^a(t',p). \tag{47}$$

Here $\tilde\phi^a_{(0)}$ is the solution of the homogeneous equation

$$\hat L_4^{ab}(t,p)\tilde\phi^b(t,p)=0\,, \quad \tilde\phi^a_{(0)}(t,p)=(1+t|p|)\mathrm{e}^{-t|p|}\tilde\phi^a_{(0)}(p)\,. \tag{48}$$

The Green's function

$$\hat L_4^{ab}(t,p)G_p(t,t')=\delta(t-t') \tag{49}$$

can easily be found:

$$G_p(t,t')=\frac{1}{2|p|^3}\phi_-(t'|p|)[\phi_+(t'|p|)-\phi_-(t'|p|)]\theta(t-t') + (t\leftrightarrow t')\,, \tag{50}$$

with

$$\phi_-(x)=(1+x)\mathrm{e}^{-x}\,,\quad \phi_+(x)=(1-x)\mathrm{e}^{x}\,. \tag{51}$$

The solution of the integral equation (47) can be approached through an iterative procedure:

$$\tilde\phi^a(t,p)=\tilde\phi^a_{(0)}(t,p) + \int_0^\infty \mathrm{d}t' G_p(t,t')\big[\hat L_2^{ab}(t',p)+\hat L_1^{ab}(t',p)\big]\tilde\phi^a_{(0)}(t',p) + \text{negligible terms}\,. \tag{52}$$

Expanding now the result in a t-power series one can see that the neglected terms in the above equation are of order $\mathcal{O}(t^4)$ and thus are irrelevant for our purposes. The symmetric part of the solution (52) is easily determined to be

$$\left[1-\frac12|p|^2t^2-\frac13 t^3(\mathbf{f}^2|p|+\mathrm{i}\mathbf{f}\cdot\mathbf{f}'\,\mathrm{sign}\,p+\mathbf{f}\cdot\mathbf{g})\right]\tilde\phi^a_{(0)}(p) + O(t^4)\,, \tag{53}$$

while the contribution to the antisymmetric part is

$$\int_0^\infty \mathrm{d}t' G_p(t,t')\left(\frac{1}{t'}-\partial_{t'}\right)\mathrm{e}^{-|p|t'}(1+|p|t')r^{ab}\tilde\phi^a = -\frac13 t^3\left[\Gamma(0,2|p|t)+\frac{25}{12}\right]r^{ab}\tilde\phi^a+O(t^4)\,. \tag{54}$$

The next step is to integrate the 'annoying' incomplete gamma function:

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}ph}\Gamma(0,2t|p|) = 2\,\mathrm{Re}\lim_{\varepsilon\to0}\int_0^\infty \mathrm{d}p\mathrm{e}^{\mathrm{i}ph}\Gamma(\varepsilon,2t|p|) = 2\,\mathrm{Re}\lim_{\varepsilon\to0}\frac{t}{2\mathrm{i}h}\Gamma(\varepsilon)\left[1-\frac{1}{\left(1+\frac{\mathrm{i}h}{2t}\right)^{\varepsilon}}\right] = \frac1t+\mathcal{O}(h)\,, \tag{55}$$

and thus the $\mathcal{O}(t^3)$ antisymmetric contribution to the solution can be taken to be just

$$-\frac13 t^3\frac{25}{12}r^{ab}=-\frac13 t^3\kappa(g^af^b-g^bf^a)\,. \tag{56}$$

To obtain the final result one must take into account that normal variations do not preserve the static gauge and, therefore, a redefinition of the t variable is needed. Repeating the relevant calculation of [15], we arrive at the following key result for the normal variations of the components of the **g**-function:

$$\frac{\delta g^a(s)}{\delta\mathbf{n}^b(s')}=\int\frac{\mathrm{d}p}{2\pi}[|p|^3-|p|(\mathbf{f}^2\delta^{ab}-3f^af^b)]\mathrm{e}^{\mathrm{i}ph} - \left[\mathbf{f}\cdot\mathbf{g}\delta^{ab}-\frac32(f^ag^b+f^bg^a) + \kappa(f^ag^b-f^bg^a)\right]\delta(h)+\mathcal{O}(h)\,. \tag{57}$$

It should be stressed, at this point, that the arbitrariness of the number κ appearing in (56) and (57) is related to the arbitrary number κ_0, which appears in (40) by $\kappa=\frac{25}{12}\kappa_0$. The origin of this arbitrariness is the fact that one cannot define uniquely an orthonormal basis on the 5-dimensional surface.[2]

4 Loop equation and Bianchi identity

Beginning this section we perform a first check of (22) by using it to verify the Makeenko–Migdal (MM) equation [10], see also the extensive review expositions in [11, 12], for *differentiable*, non-self-intersecting Wilson loops that are traversed only once, namely

$$\tilde\Delta W[C]\approx 0\,, \tag{58}$$

where the symbol $\approx$ means that the finite part on the r.h.s. is zero and the MM loop operator is defined in [11, 12] as

$$\tilde\Delta=\oint_C \mathrm{d}c_\nu\partial^c_\mu\frac{\delta}{\delta\sigma_{\mu\nu}(c)} = \lim_{\eta\to0}\lim_{\eta'\to0}\int\mathrm{d}s\,c'_\nu(s)\int_{s-\eta}^{s+\eta}\mathrm{d}s'\frac{\delta}{\delta c_\mu(s')} \times\int_{-\eta'}^{\eta'}\mathrm{d}hh\frac{\delta^2}{\delta c_\mu(s+h)\delta c_\nu(s)}\,. \tag{59}$$

It can, now, be easily determined from (22) that

$$\tilde\Delta A_{\min}=2\lim_{\eta\to0}\int\mathrm{d}s\,c'_\nu(s)\int_{s-\eta}^{s+\eta}\mathrm{d}s'\frac{\delta}{\delta c_\mu(s')}[t_\nu(s)g_\mu(s)] = 2\lim_{\eta\to0}\int\mathrm{d}s\int_{s-\eta}^{s+\eta}\mathrm{d}s'\frac{\delta g_\mu(s)}{\delta c_\mu(s')}\,. \tag{60}$$

[2] The freedom of choosing of such a basis was ignored in a previous work, namely [21], where κ_0 was arbitrarily set to 1.

From (18) we obtain

$$\frac{\delta g_\mu(s)}{\delta c_\nu(s')} = \frac{\delta g^a(s)}{\delta \mathbf{n}^b(s')} n^a_\mu(s) n^b_\nu(s') - R_{\mu\nu}(s,s')\delta'(s-s') - g_\mu(s) t_\nu(s) \delta'(s-s') \,. \tag{61}$$

One can easily check that $R'_{\mu\mu}(s,s)=0$ and consequently

$$\tilde{\Delta} A_{\min} = 2 \lim_{\eta\to 0} \int \frac{\delta g^a(s)}{\delta \mathbf{n}^b(s')} \mathbf{n}^a(s)\cdot \mathbf{n}^b(s') \,. \tag{62}$$

From (57) we see that

$$\begin{aligned} &\frac{\delta g^a(s)}{\delta \mathbf{n}^b(s')} \mathbf{n}^a(s)\cdot \mathbf{n}^b(s') \\ &\quad = -(D-4)\mathbf{f}\cdot\mathbf{g}\delta(s-s') \\ &\qquad + \left[\frac{3!}{\pi}\frac{\delta^{ab}}{(s-s')^4} + \frac{1}{\pi}\frac{1}{(s-s')^2}(\mathbf{f}^2\delta^{ab} - 3f^a f^b)\right] \\ &\qquad \times \mathbf{n}^a(s)\cdot\mathbf{n}^b(s') + \mathcal{O}(s-s') \,, \end{aligned} \tag{63}$$

and so, in 4-dimensional space,

$$\tilde{\Delta} A_{\min} \equiv 0 \,. \tag{64}$$

It is obvious from the derivation of the above result that we do not need to know the antisymmetric part of the normal deviations of the **g**-function for the verification of the MM loop equation. This means that the fact that the numerical value of κ is unknown is of no importance, as far as the verification of the loop equation is concerned. By juxtaposition, for the verification of the Bianchi identity the antisymmetric part of (57) plays a crucial role as we shall now witness.

To this end let us refer to (18), through which we find that

$$\begin{aligned} &t_\mu(s)\frac{\delta g_\nu(s)}{\delta c_\lambda(s')} - (\mu \leftrightarrow \nu) \\ &\quad = \frac{\delta g^a(s)}{\delta \mathbf{n}^b(s')} n^b_\lambda(s') t_{[\mu}(s) n^a_{\nu]}(s) \\ &\qquad + \delta'(s-s')\mathbf{t}(s)\cdot\mathbf{n}^a(s') n^a_\lambda(s') t_{[\mu}(s) g_{\nu]}(s) \,, \end{aligned} \tag{65}$$

which finally gives

$$\begin{aligned} &\epsilon^{\kappa\lambda\mu\nu}\partial^{c(s)}_\lambda \frac{\delta A_{\min}}{\delta\sigma_{\mu\nu}(c(s))} \\ &\quad = \epsilon^{\kappa\lambda\mu\nu} \lim_{\eta\to 0}\int_{s-\eta}^{s+\eta} \mathrm{d}s' \frac{\delta g^a(s)}{\delta \mathbf{n}^b(s')} n^b_\lambda(s') t_{[\mu}(s) n^a_{\nu]}(s) \\ &\qquad + \epsilon^{\kappa\lambda\mu\nu}\mathbf{t}(s)\cdot\mathbf{n}'^a(s) n^a_\lambda t_\mu(s) g_\nu(s) \,. \end{aligned} \tag{66}$$

One observes that in the first term of the above equation only the antisymmetric part of the normal variation of the **g**-function survives. As far as the second term is concerned, we can use the arguments presented in the previous section to write $\mathbf{n}'^a = -(\mathbf{n}^a\cdot\mathbf{f})\mathbf{t}$. The result expressed by (57) leads us now to conclude that

$$\epsilon^{\kappa\lambda\mu\nu}\partial^{c(s)}_\lambda \frac{\delta A_{\min}}{\delta\sigma_{\mu\nu}(c(s))} = (2\kappa - 1)\epsilon^{\kappa\lambda\mu\nu} f_\lambda(s) t_{[\mu}(s) g_{\nu]}(s) \,. \tag{67}$$

At this point, κ enters as an arbitrary constant, rendering the Bianchi identity conditional. As now becomes apparent from (27), (37) and (40), the arbitrariness of this constant refers to the fact that we cannot connect uniquely the orthonormal basis $\{n^a_M(t,s)\}$, defined on the surface, with the orthonormal basis $\{n^a_\mu(s), t_\mu(s)\}$ defined on the boundary. It is important to realize at the same time that if the **g**-function were known, one could, in principle, compute its normal variations unambiguously.

In the next section, we explicitly determine the normal variations of the **g**-function for the non-trivial as well as generic smooth (Wilson) contour configuration discussed in [15], which goes by the name of the 'wavy line' configuration. As we shall see, the explicit result determines the constant κ to be $1/2$, as it bypasses the need of referring to a choice of basis, $\{n^a_M(t,s)\}$, of the form employed in the analysis in Sect. 3 and leading to the result expressed by (67). Given, now, that κ, as was introduced in this section, does not depend on the specific form of the (smooth) Wilson loop boundary, we consider the relevant result to be an independent way to determine the value of κ.

5 Wavy line Wilson contour and the Bianchi identity

The wavy line approximation [15] is specified by the assumption that the closed Wilson contours entering the gauge field–string duality are described by

$$c_1(s) = s\,, \quad c_i = \phi_i(s)\,, \quad i = 2,\ldots,D\,, \tag{68}$$

with the transverse components $\phi_i(s)$ being very small. Our objective in this section is to expand, to fourth order, $A_{\min}$ in powers of the ϕ_i. Following [15], we begin with the Hamilton–Jacobi equation for the minimal surface, which for $y(s) = y \to 0$ can be written as

$$\begin{aligned} &\frac{\partial A_{\min}}{\partial y} \\ &\quad = -\frac{1}{y^2}\int \mathrm{d}s \sqrt{\mathbf{c}'^2 - y^4\left(\frac{\delta A_{\min}}{\delta \mathbf{c}(s)}\right)^2} \\ &\quad = -\frac{1}{y^2}\int \mathrm{d}s \sqrt{\mathbf{c}'^2 - y^4\left(\frac{\delta A_{\min}}{\delta \boldsymbol{\phi}(s)}\right)^2 - y^4\left(\boldsymbol{\phi}\cdot\frac{\delta A_{\min}}{\delta \boldsymbol{\phi}(s)}\right)^2} \,, \end{aligned} \tag{69}$$

where, for the last step, we used reparametrization invariance:

$$\mathbf{c}'\cdot\frac{\delta A_{\min}}{\delta \mathbf{c}(s)} = 0 \,. \tag{70}$$

To continue we now assume that the minimal area can be cast into the following general form:

$$A_{\min} = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}s_1\ldots\mathrm{d}s_n \Gamma_{i_1\ldots i_n}(s_1,\ldots,s_n|y) \times \phi_{i_1}(s_1)\cdots\phi_{i_n}(s_n) \,. \tag{71}$$

Inserting the above equation into (69), expanding the square root and taking the Fourier transform of both sides, one finds

$$\begin{aligned} A_{\min} = {} & \frac{L_0}{y} + \frac{1}{2}\int \frac{\mathrm{d}p}{2\pi}\tilde{\Gamma}_2(p|y)\tilde{\phi}_i(p)\tilde{\phi}_i(-p) \\ & + \frac{1}{8}\int \frac{\mathrm{d}p_1}{2\pi}\cdots\frac{\mathrm{d}p_4}{2\pi}\tilde{\Gamma}_4(p_1,p_2,p_3,p_4|y) \\ & \times \tilde{\phi}_i(p_1)\tilde{\phi}_i(p_2)\tilde{\phi}_j(p_3)\tilde{\phi}_j(p_4)2\pi\delta\left(\sum_{i=1}^{4}p_i\right) + \mathcal{O}(\phi^6)\,. \end{aligned} \tag{72}$$

In the above expression L_0 is the length of the contour (along the direction 1) and we have written

$$\begin{aligned} & \Gamma_{i_1i_2}(s_1,s_2|y) \\ & \quad = \delta_{i_1i_2}\Gamma_2(p|y) \\ & \quad = \delta_{i_1i_2}\int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}p(s_2-s_1)}\tilde{\Gamma}_2(p|y)\,, \\ & \Gamma_{i_1i_2i_3i_4}(s_1,s_2,s_3|y) \\ & \quad = (\delta_{i_1i_2}\delta_{i_3i_4} + \text{perms})\Gamma_4(s_2-s_1,s_3-s_1,s_4-s_1|y)\,, \\ & \Gamma_4(s_2-s_1,s_3-s_1,s_4-s_1|y) \\ & \quad = \int\frac{\mathrm{d}p_1}{2\pi}\cdots\frac{\mathrm{d}p_4}{2\pi}2\pi\delta\left(\sum_{i=1}^{4}p_i\right) \\ & \qquad \times\exp\left(\mathrm{i}\sum_{i=1}^{4}p_is_i\right)\tilde{\Gamma}_4(p_1,p_2,p_3,p_4|y)\,. \end{aligned} \tag{73}$$

The functions $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_4$ have been determined in [15]. Here we present only the leading, finite part of their expansion in powers of y:

$$\tilde{\Gamma}_2 = -|p|^3\,, \tag{74}$$

$$\begin{aligned} \tilde{\Gamma}_4 = {} & \Phi(p_1,p_3)+\Phi(p_1,p_4)+\Phi(p_2,p_3)+\Phi(p_2,p_4) \\ & -\Phi(p_1,p_2)-\Phi(p_3,p_4)-F(p_1,p_2,p_3,p_4|y)\,, \end{aligned} \tag{75}$$

with

$$\begin{aligned} F = {} & \left[2\frac{\epsilon_{p_1}\epsilon_{p_2}\epsilon_{p_3}\epsilon_{p_4}+1}{\Delta^3} + \frac{\epsilon_{p_1}\epsilon_{p_2}\epsilon_{p_3}\epsilon_{p_4}}{\Delta^2}\left(\sum_{i=1}^{4}\frac{1}{|p_i|}\right)\right. \\ & \left. + \frac{\sum_{i<j}|p_ip_j|}{\Pi\Delta} - \frac{\Delta}{\Pi}\right]\Pi^2\,, \\ \Phi(p_1,p_2) = {} & \left[2\frac{\epsilon_{p_1}\epsilon_{p_2}}{\Delta^3} + \frac{\epsilon_{p_1}\epsilon_{p_2}}{\Delta^2}\left(\frac{1}{|p_1|}+\frac{1}{|p_2|}\right) + \frac{1}{\Delta}\frac{1}{p_1p_2}\right]\Pi^2 \end{aligned} \tag{76}$$

and

$$\epsilon_p = \operatorname{sign}p\,,\quad \Delta = \sum_{i=1}^{4}|p_i|\,,\quad \Pi = p_1p_2p_3p_4\,. \tag{77}$$

Given the above relations our first check will refer to the normal variations of the **g**-function. In particular, we shall prove that no term $\sim\delta'(s_1-s_2)$ appears in the transverse variation of the **g**-function and that the coefficient of the antisymmetric part is $\frac{1}{2}$. The quantity of interest reads

$$\begin{aligned} \frac{\delta g^a(s_1)}{\delta\mathbf{n}^b(s_2)} & = n_\mu^a(s_1)n_\nu^b(s_2)\frac{\delta g_\mu(s_1)}{\delta c_\nu(s_2)} \\ & = n_i^a(s_1)n_j^b(s_2)\left(\phi_i(s_1)\phi_j(s_2)\frac{\delta g_i(s_1)}{\delta c_1(s_2)}\right. \\ & \quad \left. -\phi_i(s_1)\frac{\delta g_1(s_1)}{\delta c_j(s_2)} - \phi_j(s_2)\frac{\delta g_i(s_1)}{\delta c_1(s_2)} + \frac{\delta g_i(s_1)}{\delta c_j(s_2)}\right)\,, \end{aligned} \tag{78}$$

where we have taken account of the fact that $c'_\mu n_\mu^a = 0 \Rightarrow n_1^a = -\phi'_i n_i^a$. It should also be noted that in the preceding equation we have written $s_1 = s+\frac{h}{2}$ and $s_2 = s-\frac{h}{2}$ and for convenience we shall eventually integrate both sides over s.

Using now reparametrization invariance, we write

$$g_1 = -\phi'_i g_i = \frac{1}{\sqrt{\mathbf{c}'^2}}\phi'_i\frac{\delta A_{\min}}{\delta\phi_i}\,,\quad \frac{\delta A_{\min}}{\delta c_1} = -\phi'_i\frac{\delta A_{\min}}{\delta\phi_i}\,. \tag{79}$$

Substituting (79) into (78) and keeping terms up to second order, we find

$$\begin{aligned} \frac{\delta g^a(s_1)}{\delta\mathbf{n}^b(s_2)} = {} & n_i^a(s_1)n_j^b(s_2)\left(\delta'(s_1-s_2)A_{ij} - \frac{\delta^2A_{\min}^{(4)}}{\delta\phi_i(s_1)\delta\phi_j(s_2)}\right) \\ & + n_i^a(s_1)n_j^b(s_2)\Sigma_{ij} + \mathcal{O}(\phi^4)\,, \end{aligned} \tag{80}$$

where

$$\begin{aligned} A_{ij} = {} & (\phi'_j(s_1)-\phi'_j(s_2))\frac{\delta A_{\min}^{(2)}}{\delta\phi_i(s_1)} + \phi'_j(s_2)\frac{\delta A_{\min}^{(2)}}{\delta\phi'_i(s_2)} \\ & -\phi'_i(s_1)\frac{\delta A_{\min}^{(2)}}{\delta\phi_j(s_1)} \end{aligned} \tag{81}$$

and

$$\begin{aligned} \Sigma_{ij} = {} & \frac{1}{2}\phi'_k(s_1)\phi'_k(s_1)\frac{\delta^2A_{\min}^{(2)}}{\delta\phi_i(s_1)\delta\phi_j(s_2)} \\ & -\phi'_i(s_1)\phi'_k(s_1)\frac{\delta^2A_{\min}^{(2)}}{\delta\phi_k(s_1)\delta\phi_j(s_2)} \\ & -\phi'_j(s_2)\phi'_k(s_2)\frac{\delta^2A_{\min}^{(2)}}{\delta\phi_i(s_1)\delta\phi'_k(s_2)}\,. \end{aligned} \tag{82}$$

In the above equations the expressions $A_{\min}^{(2)}$ and $A_{\min}^{(4)}$ refer to the minimal area estimation up to second and fourth order, respectively, and they can be read from (72). As we are interested only in the antisymmetric part of the normal variations (80), we shall ignore the contribution from the term (82), since it is purely symmetric. It is, now, easy to determine that

$$\frac{\delta^2A_{\min}^{(2)}}{\delta\tilde{\phi}_i(k)\delta\tilde{\phi}_j(k')} = \delta_{ij}2\pi\delta(k+k')\tilde{\Gamma}_2(k) \tag{83}$$

and

$$\begin{aligned}&\frac{\delta^2 A^{(4)}_{\min}}{\delta\tilde{\phi}_i(k)\delta\tilde{\phi}_j(k')}\\&=\int\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}2\pi\delta(p_1+p_2+k+k')\\&\quad\times\left(\tilde{M}(p_1,p_2,k,k')+\frac{1}{2}\tilde{\Gamma}_4(p_1,p_2,k,k')\delta_{ij}\right)\tilde{\phi}_i(p_1)\tilde{\phi}_j(p_2)\\&\quad+\int\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}2\pi\delta(p_1+p_2+k+k')\\&\quad\times\tilde{\Lambda}(p_1,p_2,k,k')\tilde{\phi}_i(p_1)\tilde{\phi}_j(p_2)\,,\end{aligned}\tag{84}$$

with

$$\tilde{M}\equiv\Phi(p_1,p_2)+\Phi(k,k')-F(p_1,p_2,k,k')\tag{85}$$

and

$$\tilde{\Lambda}\equiv\Phi(k,p_1)+\Phi(k',p_2)-\Phi(k,p_2)-\Phi(k',p_1)\,.\tag{86}$$

Taking the Fourier transform of (83) we find

$$\begin{aligned}\frac{\delta^2 A^{(2)}_{\min}}{\delta\phi_i(s_1)\delta\phi_j(s_2)}&=\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}k'}{2\pi}\mathrm{e}^{-\mathrm{i}ks_1-\mathrm{i}k's_2}\frac{\delta^2 A^{(2)}_{\min}}{\delta\tilde{\phi}_i(k)\delta\tilde{\phi}_j(k')}\\&=-\delta_{ij}\int\frac{\mathrm{d}k}{2\pi}|k|^3\mathrm{e}^{-\mathrm{i}k(s_1-s_2)}\,,\end{aligned}\tag{87}$$

and consequently

$$\begin{aligned}\frac{\delta^2 A^{(2)}_{\min}}{\delta\phi_i(s)\delta\phi_j(s_2)}&=\int\mathrm{d}s'\,\Gamma_2(s-s')\phi_i(s')\,,\\\Gamma_2(s)&=-\int\frac{\mathrm{d}k}{2\pi}|k|^3\mathrm{e}^{-\mathrm{i}ks}\,.\end{aligned}\tag{88}$$

One now observes that only the last term on the r.h.s. of (84) gives an antisymmetric contribution, so the first one can be ignored. Employing once again the Fourier transform in (84), one sees that

$$\begin{aligned}&\frac{\delta^2 A^{(4)}_{\min}}{\delta\phi_i\left(s+\frac{h}{2}\right)\delta\phi_j\left(s-\frac{h}{2}\right)}\\&\quad=\int\frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_2}{2\pi}2\pi\delta(p_1+p_2+q)\\&\qquad\times\mathrm{e}^{-\mathrm{i}qs-\mathrm{i}hk}\tilde{\Lambda}\left(p_1,p_2,k+\frac{q}{2},-k+\frac{q}{2}\right)\tilde{\phi}_i(p_1)\tilde{\phi}_j(p_2)\,.\end{aligned}\tag{89}$$

Since we are interested in the limit $|h|\to 0$, we shall explore the limit $|k|\to\infty$ in the above relation. As pointed out already, it is enough for our purposes to examine the version of (78) integrated over s, so we can consider the case $q=0$, $p_1=-p_2\equiv p$ in the last relation.

Using (76) and (86) we determine

$$\begin{aligned}\tilde{\Lambda}(p,-p,k,-k)&=4\Phi(p,k)\\&=4\left[\frac{\epsilon_p\epsilon_k}{4(|p|+|k|)^3}+\frac{\epsilon_p\epsilon_k}{4(|p|+|k|)^2}\left(\frac{1}{|p|}+\frac{1}{|k|}\right)\right.\\&\quad\left.+\frac{1}{2(|p|+|k|)pk}\right]p^4k^4\\&=\epsilon_p\epsilon_k|p|^5\left[\frac{x^4}{(1+x)^3}+\frac{3x^3}{1+x}\right],\end{aligned}\tag{90}$$

where [15] we have set $x=\frac{|k|}{|p|}$. Upon taking the limit $x\to\infty$ we find that

$$\begin{aligned}\tilde{\Lambda}(p,-p,k,-k)&=\epsilon_p\epsilon_k|p|^5\left[3x^2-2x+\mathcal{O}\left(\frac{1}{x}\right)\right]\\&=3p^3k^2\,\mathrm{sign}\,k-2p|p|^3k+\mathcal{O}\left(\frac{1}{k}\right).\end{aligned}\tag{91}$$

The first term gives a zero contribution in the limit $h\to 0$, while the second one leads to

$$\begin{aligned}&\int\mathrm{d}s\frac{\delta^2 A^{(4)}_{\min}}{\delta\phi_i(s+h/2)\delta\phi_i(s-h/2)}\\&=\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-\mathrm{i}hk}\tilde{\Lambda}(p,-p,k,-k)\tilde{\phi}_i(p)\tilde{\phi}_j(-p)\\&=-2\mathrm{i}\delta'(h)\int\frac{\mathrm{d}p}{2\pi}|p|^3\tilde{\phi}_i(p)\tilde{\phi}_j(-p)\\&=\delta'(h)\int\mathrm{d}s\,\mathrm{d}s'[\phi_i'(s)\phi_j(s')-\phi_i(s)\phi_j'(s')]\Gamma_2(s-s')\\&=-\delta'(h)\int\mathrm{d}s\left[\phi_j'(s)\frac{\delta A^{(2)}_{\min}}{\delta\phi_i(s)}-\phi_i'(s)\frac{\delta A^{(2)}_{\min}}{\delta\phi_j(s)}\right].\end{aligned}\tag{92}$$

This term exactly cancels the term that appears in (80) in the limit $h\to 0$. Thus, it is confirmed, in the framework of the wavy line approximation, that no term $\propto\delta(h')$ appears in the transverse variation of the **g**-function. The first term in (81) reads, in the limit $h\to 0$,

$$\begin{aligned}(\phi_j'(s_1)-\phi_j'(s_2))\frac{\delta A^{(2)}_{\min}}{\delta\phi_i(s_1)}&=h\phi_j''(s)\frac{\delta A^{(2)}_{\min}}{\delta\phi_i(s)}+\mathcal{O}(h^2)\\&=-h\phi_j''(s)g_i(s)+\mathcal{O}(h^2)+\mathcal{O}(\phi^4).\end{aligned}\tag{93}$$

Thus, the antisymmetric part of the transverse variation reads

$$-\frac{1}{2}n_i^a n_j^b(\phi_i''g_j-\phi_j''g_i)\,,\tag{94}$$

which leads to the conclusion that the value of the constant κ that appears in (67) of Sect. 5 is $1/2$. As this constant is independent from the details of the contour that forms the boundary, we consider the result (94) as valid for an arbitrary contour and thereby establish the validity of the Bianchi identity, equivalently zig zag invariance, for the string–gauge field connection scenario promoted in [7] by Polyakov.

6 Concluding remarks

In this work, we have verified a property, which is important from the standpoint of physics, of the Wilson loop functional in the framework of the AdS/CFT – as promoted in [7] in the $\lambda \to \infty$ limit and concretely deliberated in [15, 16]. In particular, we established a condition for the validity of the Bianchi identity, which, in turn, solidifies the consistency of the string–gauge field connection in the sense that it is compatible with zig zag invariance and hence secures the validation of the Stokes theorem. This very important issue has been explicitly demonstrated in the context of the wavy line approximation, which sufficiently describes, in a general manner, a smooth Wilson loop contour. From the physics point of view, what we find especially worth noting is that the results in this paper have been obtained without any knowledge of the **g**-function. The latter is expected to carry all the dynamics in any particular investigation of interest one wishes to conduct in the context of the string-based theoretical scheme adopted in this work. Given, now, that string theory per se is formulated in the framework of first quantization, it seems realistic to further pursue the issue of the string–gauge field relation by employing first quantization methodologies on the field side. The strategy we specifically have in mind to apply for pursuing such a connection would involve, on the side of gauge field theory, a first quantization, worldline casting of gauge field systems, with which we happen to be quite familiar (see, e.g., [20] for a typical example). The envisioned focus of attention in such a study is expected to be placed on the **g**-function in the sense of connecting it with (non-perturbative) dynamical behaviors in gauge field systems. Preliminary indications seem to point to a direction according to which the **g**-function is directly linked with the spin–field interaction dynamics, while perturbative (local) dynamics are associated to the formation of cusps on the Wilson contour. Such speculations are, of course, subject of concrete scrutiny, which we intend to explore in the immediate future.

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